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Two particles with opposite charge in a homogeneous magnetic field: particular analytical solutions of the two-dimensional Schrödinger equation

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Abstract. The two-dimensional Schrödinger equation for two particles with opposite charge and Coulomb interaction in a homogeneous magnetic field (perpendicular to the plane) is investigated. Analytical solutions are found for zero quasi-momentum (pair at rest) and for an infinite, but countable set of magnetic field values.

1. Introduction

In [1] we investigated the solutions of the Schrödinger equation for two identical particles (electrons) in a homogeneous magnetic field. It was found that they form bound states despite the repulsive character of the interaction. Bound states means that the stationary solutions are normalizable and the eigenvalue spectrum is discrete. Of course, there is no binding energy (the total energy is positive definite), but the electrons are localized in a finite region. Moreover, the Hamiltonian decouples in centre of mass $\boldsymbol{R}$ and relative coordinates $\boldsymbol{r}$, the eigenfunctions factorize and for an infinite and countable set of magnetic field values the exact eigensolutions can be given analytically. This holds for ground- and excited- as well as singlet and triplet states.

In this paper we investigate the two-dimensional Schrödinger equation for two particles of opposite charge, which behave quite differently. First, the Hamiltonian does *not* decouple if expressed by $\boldsymbol{R}$ and $\boldsymbol{r}$ and the total spectrum is continuous [2]. The centre of mass can move with a certain quasimomentum in space, roughly speaking, because the centre of mass has no net charge. This degree of freedom is responsible for the continuity of the spectrum. On the other hand, there are discrete internal excitation energies which can be calculated analytically under certain circumstances (see below). Qualitatively, the behaviour of the second case agrees with a neutral atom in a magnetic field (see e.g. [3]).

The different behaviour of the two cases can be demonstrated in an suggestive way by means of a typical special classical trajectory in either case (see figure 1). (The *general* classical description is more complicated and is not the subject of this paper.) For this introductory consideration we assume both masses to be equal. For *equal* charges the particles can orbit on the same circle (radius R) with the same modulus of velocity v and with a phase difference of

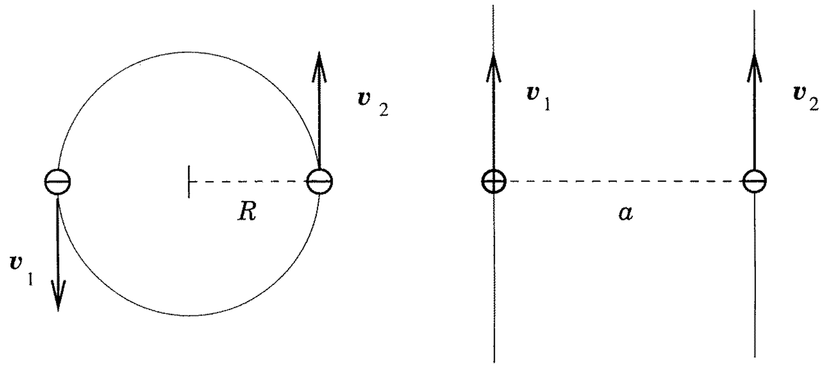

Figure 1. Special orbits for two particals of like (left) and opposite (right) charge.

π. From the equilibrium condition for all forces (Lorenz, centrifugal, and Coulomb force) we obtain

$$R = \frac{R_c}{2} + \frac{R_c}{2}\sqrt{1 + \frac{e^2}{m}\frac{\omega_c}{v^3}}$$

where the cyclotron radius $R_c = \frac{mc}{e}\frac{v}{B}$ and the cyclotron frequency $\omega_c = \frac{e}{mc}B$ (both in the one-particle case) have been introduced. Interestingly, If we solve the equilibrium condition for v, we obtain

$$v = \frac{1}{2}\omega_c R \pm \frac{1}{2}\omega_c R\sqrt{1 - \frac{e^2}{m}\frac{1}{\omega_c^2 R^3}}.$$

Thus for the same orbit there are two possible velocities and for a given magnetic field there is no orbit with $R < R_{min} = (\frac{e^2}{m\omega_c^2})^{1/3}$. Likewise, the frequency $\omega = \frac{v}{R}$ is a two-valued function of R which exists only for $R > R_{min}$. Remember, that the cyclotron frequency for a single particle does *not* depend on the radius at all.

For *opposite* charges an orbit as described above does not exist, mainly because the *noninteracting* particles would circle with opposite sense of rotation. In this case, the simplest orbit consists of two parallel straight lines on which the particles move in the same direction with the same velocity and in phase (i.e. the difference vector $\boldsymbol{r}_2 - \boldsymbol{r}_1$ is perpendicular to the orbits). From the equilibrium condition it follows for the distance of the two lines $a = \sqrt{\frac{ec}{vB}}$ or for the velocity $v = \frac{ec}{a^2 B}$. Below, we see that these qualitative differences between equal and opposite charges are also apparent in the quantum mechanical treatment.

In [4] we discussed the two-dimensional hydrogen atom in an homogeneous magnetic field which is a special case of this work (for an infinite positive mass). For *harmonic* particle–particle interaction the general solution of the two-particle problem in a magnetic field has already been found [5, 6]. However, this model interaction is not very realistic. Here, we present exact solutions with the realistic interaction under some restrictions (vanishing quasimomentum, particular magnetic fields). Possible applications of these solutions might comprise electron–positron pairs, excitons, or electron–correlation hole pairs in composite Fermion systems at even denominator filling factors (see e.g. [7]).

2. Decoupling

The Hamiltonian for two particles with charges $e_1 = -e_2 = e$ and masses m_1 and m_2 reads (in cgs units)

$$H = \frac{1}{2m_1}\left(\boldsymbol{p}_1 - \frac{e}{c}\boldsymbol{A}_1\right)^2 + \frac{1}{2m_2}\left(\boldsymbol{p}_2 + \frac{e}{c}\boldsymbol{A}_2\right)^2 - \frac{e^2}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|} + H_{spin} \tag{1}$$

where the spin contribution $H_{spin} = g(\boldsymbol{s}_1 + \boldsymbol{s}_2)\cdot\boldsymbol{B}$ is disregarded in the following because its contribution is trivial. We adopt the symmetric gauge $\boldsymbol{A}_i = \boldsymbol{A}(\boldsymbol{r}_i) = \frac{1}{2}(\boldsymbol{B}\times\boldsymbol{r}_i)$ throughout. It just needs some straightforward analysis to show that the operator of the *quasimomentum*

$$\boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2 \qquad \boldsymbol{k}_1 = \boldsymbol{p}_1 + \frac{e}{c}\boldsymbol{A}_1 \qquad \boldsymbol{k}_2 = \boldsymbol{p}_2 - \frac{e}{c}\boldsymbol{A}_2 \tag{2}$$

commutes with the Hamiltonian and its components commute with each other (the latter holds only if the pair is neutral).

Now we introduce relative and centre of mass coordinates

$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1 \qquad \boldsymbol{R} = \frac{m_1}{M}\boldsymbol{r}_1 + \frac{m_2}{M}\boldsymbol{r}_2 \tag{3}$$

which give rise to the definition of new momentum operators

$$\boldsymbol{p} = \frac{\hbar}{\mathrm{i}}\boldsymbol{\nabla}_r = \frac{\mu}{m_2}\boldsymbol{p}_2 - \frac{\mu}{m_1}\boldsymbol{p}_1 \qquad \boldsymbol{P} = \frac{\hbar}{\mathrm{i}}\boldsymbol{\nabla}_R = \boldsymbol{p}_1 + \boldsymbol{p}_2 \tag{4}$$

and the vector potential in the new variable reads

$$\boldsymbol{A}_r = \boldsymbol{A}_2 - \boldsymbol{A}_1 \qquad \boldsymbol{A}_R = \frac{m_1}{M}\boldsymbol{r}_1 + \frac{m_2}{M}\boldsymbol{r}_2 \tag{5}$$

where the total mass $M = m_1 + m_2$ and the reduced mass $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$ have been introduced. The Hamiltonian in the new variables

$$\begin{aligned} H(\boldsymbol{r},\boldsymbol{R}) = {} & \frac{1}{2M}\boldsymbol{P}^2 + \frac{1}{2\mu}\left(\frac{e}{c}\right)^2\boldsymbol{A}_R^2 \\ & + \frac{1}{2\mu}\boldsymbol{p}^2 + \frac{\mu^2}{2}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right)\left(\frac{e}{c}\right)^2\boldsymbol{A}_r^2 + \frac{(m_1 - m_2)}{m_1 m_2}\left(\frac{e}{c}\right)\boldsymbol{A}_r\cdot\boldsymbol{p} - \frac{e^2}{r} \\ & + \frac{(m_1 - m_2)}{m_1 m_2}\left(\frac{e}{c}\right)^2\boldsymbol{A}_R\cdot\boldsymbol{A}_r + \frac{1}{M}\left(\frac{e}{c}\right)\boldsymbol{A}_r\cdot\boldsymbol{P} + \frac{1}{\mu}\left(\frac{e}{c}\right)\boldsymbol{A}_R\cdot\boldsymbol{p} \end{aligned} \tag{6}$$

is not yet decoupled (unlike the case of the particles with equal charge [1]). In these new variables, the quasimomentum operator reads

$$\boldsymbol{k} = \boldsymbol{P} - \frac{e}{c}\boldsymbol{A}_r. \tag{7}$$

Now, because of their commutation $[H, \boldsymbol{k}] = 0$, the common eigenfunctions of $\boldsymbol{k}$ and H can be written in the form (see e.g. [3])

$$\Psi_{\boldsymbol{\kappa}}(\boldsymbol{r},\boldsymbol{R}) = \frac{1}{\sqrt{A}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\boldsymbol{\kappa} + \frac{e}{c}\boldsymbol{A}_r)\cdot\boldsymbol{R}}\phi_{\boldsymbol{\kappa}}(\boldsymbol{r}) \tag{8}$$

where $\boldsymbol{\kappa}$ is the eigenvalue of the operator $\boldsymbol{k}$ and $\phi_{\boldsymbol{\kappa}}(\boldsymbol{r})$ has to fulfil the internal Schrödinger equation

$$H_{\boldsymbol{\kappa}}(\boldsymbol{r})\phi_{\boldsymbol{\kappa}}(\boldsymbol{r}) = E_{\boldsymbol{\kappa}}\phi_{\boldsymbol{\kappa}}(\boldsymbol{r}) \tag{9}$$

with the internal Hamiltonian

$$H_{\boldsymbol{\kappa}}(\boldsymbol{r}) = \frac{1}{2\mu}\left[\boldsymbol{p} + \frac{(m_1 - m_2)}{M}\frac{e}{c}\boldsymbol{A}_r\right]^2 + \left[\frac{\boldsymbol{\kappa}}{2} + \frac{e}{c}\boldsymbol{A}_r\right]^2 - \frac{e^2}{r} \tag{10}$$

and the normalization condition (in the area A).

$$\int_A \mathrm{d}^2\boldsymbol{r}\,|\phi_{\boldsymbol{\kappa}}(\boldsymbol{r})|^2 = 1. \tag{11}$$

3. Solution of the internal Schrödinger equation

For the solution of equation (9) we closely follow the spirit of [1, 4]. After introducing polar coordinates (r, α), expanding the squared brackets and assuming that (without loss of generality) $\boldsymbol{\kappa} \times \boldsymbol{B}$ points in the x-direction, we obtain

$$H_{\kappa}(r,\alpha) = \frac{1}{2\mu}\left[-\overline{h}^2 r^{-1/2}\frac{\partial^2}{\partial r^2}r^{1/2} - \frac{\overline{h}^2}{r^2}\left(\frac{\partial^2}{\partial\alpha^2}+\frac{1}{4}\right) + \frac{(m_1-m_2)}{M}\frac{eB}{c}\frac{\overline{h}}{\mathrm{i}}\frac{\partial}{\partial\alpha} + \frac{(m_1-m_2)^2}{M^2}\left(\frac{eB}{2c}\right)^2 r^2\right] + \frac{2}{M}\left[\left(\frac{\kappa}{2}\right)^2 + \left(\frac{eB}{2c}\right)\kappa r\cos(\alpha) + \left(\frac{eB}{2c}\right)^2 r^2\right] - \frac{e^2}{r}. \tag{12}$$

The next step is to make the ansatz

$$\phi(r,\alpha) = \frac{\mathrm{e}^{\mathrm{i}m\alpha}}{\sqrt{2\pi}}\frac{u(r)}{\sqrt{r}} \qquad m = 0, \pm 1, \pm 2, \ldots \tag{13}$$

with the normalization condition

$$\int_0^\infty \mathrm{d}r\, |u(r)|^2 = 1 \tag{14}$$

which would satisfy all terms except the term $\propto \cos(\alpha)$. The only way to render an exact solution possible is to restrict ourselves to $\boldsymbol{\kappa} = 0$ so that the trouble-making term vanishes. Using (4), (8) and (13) it can be easily seen that in the state with $\boldsymbol{\kappa} = 0$ the *expectation value* of the total canonical momentum $\boldsymbol{P}$ and of the mechanical momentum

$$\boldsymbol{P}_{mech} = \left(\boldsymbol{p}_1 - \frac{e}{c}\boldsymbol{A}_1\right) + \left(\boldsymbol{p}_2 + \frac{e}{c}\boldsymbol{A}_2\right) = \boldsymbol{P} + \frac{e}{c}\boldsymbol{A}_r \tag{15}$$

vanish. Consequently, we confine ourselves to a pair at rest.

After some rearrangements, $u(r)$ has to fulfil the following *radial internal Schrödinger equation*

$$\left[-\frac{\overline{h}^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\overline{h}^2}{2\mu}\left(m^2 - \frac{1}{4}\right)\frac{1}{r^2} + \frac{\mu}{2}\omega_L^{*2}r^2 - \frac{e^2}{r}\right]u(r) = \left[E - \left(\frac{m_1-m_2}{m_1+m_2}\right)\overline{h}\omega_L^* m\right]u(r) \tag{16}$$

where $\omega_L^* = \frac{eB}{2\mu c}$ is the Larmor frequency with reduced mass. As to be expected, for $m_1 \to \infty$ equation (16) reduces to equation (3) in [4]. However, (16) cannot be deduced from equation (3) in [4] simply by replacing the particle mass by the reduced mass, as in the case without magnetic field. However, (16) is of the same *type* as equation (3) in [4], so that our method for the solution does not have to be repeated and the solutions of (16) can be obtained from formulae (15)–(20) in [4] by the following substitutions (after adding the the appropriate powers of $\overline{h}$, the electron mass m_e and electron charge e—because atomic units were used in [4]):

$$Z \to 1 \tag{17}$$

$$\omega_L \to \omega_L^* \tag{18}$$

$$m_e \to \mu \tag{19}$$

$$E - m\overline{h}\omega_L \to E - \left(\frac{m_1-m_2}{m_1+m_2}\right)m\overline{h}\omega_L^*. \tag{20}$$

Table 1. All *solvable* reduced Larmor frequencies ω_L^* and corresponding eigenvalues $E - (\frac{m_1-m_2}{m_1+m_2})m\hbar\omega_L^*$ for $n = 2$–10 and for (*a*) $m = 0$ and (*b*) $m = 1$ in reduced atomic units (see text). N is the number of nodes of the radial wavefunction indicating which excited state it is.

(*a*)

n	$(\omega_L^*)^{-1}$	E	N
2	0.500 000E + 00	0.400 000E + 01	1
3	0.300 000E + 01	0.100 000E + 01	2
4	0.927 200E + 01	0.431 406E + 00	3
	0.727 998E + 00	0.549 452E + 01	2
5	0.211 168E + 02	0.236 778E + 00	4
	0.388 316E + 01	0.128 761E + 01	3
6	0.403 133E + 02	0.148 834E + 00	5
	0.112 570E + 02	0.533 000E + 00	4
	0.929 632E + 00	0.645 417E + 01	3
7	0.686 380E + 02	0.101 984E + 00	6
	0.246 751E + 02	0.283 687E + 00	5
	0.468 692E + 01	0.149 352E + 01	4
8	0.107 868E + 03	0.741 648E − 01	7
	0.459 214E + 02	0.174 211E + 00	6
	0.130 953E + 02	0.610 908E + 00	5
	0.111 539E + 01	0.717 239E + 01	4
9	0.159 781E + 03	0.563 272E − 01	8
	0.767 724E + 02	0.117 230E + 00	7
	0.280 095E + 02	0.321 320E + 00	6
	0.543 732E + 01	0.165 523E + 01	5
10	0.226 154E + 03	0.442 176E − 01	9
	0.119 005E + 03	0.840 301E − 01	8
	0.512 233E + 02	0.195 224E + 00	7
	0.148 274E + 02	0.674 429E + 00	6
	0.129 016E + 01	0.775 096E + 01	5

(*b*)

n	$(\omega_L^*)^{-1}$	$E - (\frac{m_1-m_2}{m_1+m_2})\hbar\omega_L^*$	N
2	0.150 000E + 01	0.200 000E + 01	1
3	0.700 000E + 01	0.571 429E + 00	2
4	0.181 394E + 02	0.275 643E + 00	3
	0.186 059E + 01	0.268 731 6 + 01	2
5	0.366 510E + 02	0.163 707E + 00	4
	0.834 903E + 01	0.718 647E + 00	3
6	0.642 985E + 02	0.108 868E + 00	5
	0.210 161E + 02	0.333 077E + 00	4
	0.218 539E + 01	0.320 310E + 01	3
7	0.102 855E + 03	0.777 794E − 01	6
	0.415 559E + 02	0.192 512E + 00	5
	0.958 910E + 01	0.834 281E + 00	4
8	0.154 096E + 03	0.584 049E − 01	7
	0.717 176E + 02	0.125 492E + 00	6
	0.236 998E + 02	0.379 751E + 00	5
	0.248 615E + 01	0.362 005E + 01	4
9	0.219 800E + 03	0.454 960E − 01	8
	0.113 269E + 03	0.882 853E − 01	7
	0.461 803E + 02	0.216 543E + 00	6
	0.107 509E + 02	0.930 154E + 00	5
10	0.301 742E + 03	0.364 550E − 01	9
	0.167 984E + 03	0.654 824E − 01	8
	0.787 673E + 02	0.139 651E + 00	7
	0.262 373E + 02	0.419 251E + 00	6
	0.276 930E + 01	0.397 213E + 01	5

If we use reduced atomic units for expressing the results (i.e. atomic units with m_e replaced by μ)

$$\text{length} = a^*_{Bohr} = \frac{\hbar}{\mu e^2}$$
$$\text{energy} = \text{Hartree}^* = \frac{\mu e^4}{\hbar^2}$$
$$\text{frequency} = \text{Herz}^* = \frac{\mu e^4}{\hbar^3}$$

the eigensolutions for $n = 2$ (first excited states) read

$$\omega_L^* = \frac{2}{2|m| + 1} \tag{21}$$
$$E = \frac{2}{(2|m| + 1)} \left(|m| + \left(\frac{m_1 - m_2}{m_1 + m_2} \right) m + 2 \right) \tag{22}$$
$$\phi(\boldsymbol{r}) \propto \frac{\mathrm{e}^{\mathrm{i}m\alpha}}{\sqrt{2\pi}} \mathrm{e}^{-\frac{r^2}{(2|m|+1)}} \cdot r^{|m|} \cdot \left[1 - \frac{2\,r}{(2|m| + 1)} \right] \tag{23}$$

and for $n = 3$ (second excited states) we obtain

$$\omega_L^* = \frac{1}{4|m| + 3} \tag{24}$$
$$E = \frac{1}{(4|m| + 3)} \left(|m| + \left(\frac{m_1 - m_2}{m_1 + m_2} \right) m + 3 \right) \tag{25}$$
$$\phi(r) \propto \frac{\mathrm{e}^{\mathrm{i}m\alpha}}{\sqrt{2\pi}} \mathrm{e}^{-\frac{r^2}{2(4|m|+3)}} \cdot r^{|m|} \cdot \left[1 - \frac{2\,r}{(2|m| + 1)} + \frac{2\,r^2}{(2|m| + 1)(4|m| + 3)} \right]. \tag{26}$$

In our units, the essential change with respect to the results in [4] is the factor $(\frac{m_1 - m_2}{m_1 + m_2})$ in the energies. Results for fixed quantum number m and different n are shown in table 1.

For those who do not want to read [4], we give some verbal interpretation of the results. The eigenvalues E and eigenfunctions $\phi(r)$ for a given n belong to the particular (solvable) magnetic fields characterized by ω_L^*. n denotes the series of solutions and describes the number of terms in the polynomial (in brackets) in $\phi(r)$. For $n > 3$ there is more than one solution (with different node numbers) for a given n and m (see table 1).

4. Summary

We have shown that there is an infinite (but countable) set of exact analytical solutions of the Schrödinger equation for a two-dimensional system of two particles (with opposite charge) in an external homogeneous magnetic field. These solutions exist only for a certain discrete set of magnetic fields. The solutions in between these discrete field values differ from the solvable cases in that the eigenfunctions are described by infinite instead of finite polynomials. If we scan the field around a solvable value, the polynomial coefficients of *all* powers beyond some finite power vanish simultaneously, if the solvable value is met. This discreteness of our singular solutions is probably more than a mathematical curiosity, but might be somehow related to the singular behaviour of special solutions of the system of N electrons in a magnetic field, which describe the fractional quantum Hall effect. Therefore, it seems to be a certain trait of two-dimensional quantum systems in magnetic fields. However, until now it has not been clear which special physical properties may be attributed to the singular solutions given above.

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